

Mathematical Structures in Computer Science

<http://journals.cambridge.org/MSC>

Additional services for *Mathematical Structures in Computer Science*:

Email alerts: [Click here](#)

Subscriptions: [Click here](#)

Commercial reprints: [Click here](#)

Terms of use : [Click here](#)

Coalgebraic analysis of subgame-perfect equilibria in infinite games without discounting

SAMSON ABRAMSKY and VIKTOR WINSCHHEL

Mathematical Structures in Computer Science / *FirstView* Article / February 2016, pp 1 - 11
DOI: 10.1017/S0960129515000365, Published online: 23 July 2015

Link to this article: http://journals.cambridge.org/abstract_S0960129515000365

How to cite this article:

SAMSON ABRAMSKY and VIKTOR WINSCHHEL Coalgebraic analysis of subgame-perfect equilibria in infinite games without discounting. *Mathematical Structures in Computer Science*, Available on CJO 2015 doi:10.1017/S0960129515000365

Request Permissions : [Click here](#)

Coalgebraic analysis of subgame-perfect equilibria in infinite games without discounting

SAMSON ABRAMSKY[†] and VIKTOR WINSCHEL[‡]

[†]*Department of Computer Science, University of Oxford, Oxford, Oxfordshire, U.K.*
Email: samson@comlab.ox.ac.uk

[‡]*Department of Management, Technology, and Economics, ETH Zurich, Zurich, Switzerland*
Email: viktor.winschel@googlemail.com

Received September 2012; revised May 2013

We present a novel coalgebraic formulation of infinite extensive games. We define both the game trees and the strategy profiles by possibly infinite systems of corecursive equations. Certain strategy profiles are proved to be subgame-perfect equilibria using a novel proof principle of predicate coinduction which is shown to be sound. We characterize all subgame-perfect equilibria for the dollar auction game. The economically interesting feature is that in order to prove these results we do not need to rely on continuity assumptions on the pay-offs which amount to discounting the future. In particular, we prove a form of one-deviation principle without any such assumptions. This suggests that coalgebra supports a more adequate treatment of infinite-horizon models in game theory and economics.

1. Introduction

Infinite structures turn up at many places in economic theory. Prominent examples are infinite games. The standard approach is to employ discounted dynamic programming to derive equilibria (Fudenberg and Tirole 1991; Ljungqvist and Sargent 2004). In this paper we take a different approach. We shall use the methods of **coalgebra** (Jacobs and Rutten 1997; Rutten 2000) to analyse two basic examples of infinite games without resorting to discounting and induction.

We use coalgebraic methods to define infinite extensive games and strategy profiles, and the notion of **subgame-perfect equilibrium**. We then prove that certain profiles are subgame-perfect equilibria. This leads us to formulate an (apparently) novel notion of predicate coinduction, which is shown to be sound by reducing it to Kozen's metric coinduction (Kozen and Ruozi 2007). We give a complete characterization of the subgame-perfect equilibria of the dollar auction game. We also prove a version of a standard game theoretical result, the **one deviation principle**, without needing future discounting assumptions.

Our work is inspired by recent work by Pierre Lescanne (Lescanne 2011; Lescanne and Perrinel 2012), but it is explicitly coalgebraic, leading to a mathematically richer and more general approach. The coalgebraic approach we present is promising for further applications to economic modelling and we will discuss some directions for future work in the last section.

For a coalgebraic treatment of combinatorial games, see Honsell and Lenisa (2011) and Honsell *et al.* (2012).

2. Games

We shall assume familiarity with the basic elements of coalgebra (Jacobs and Rutten 1997; Rutten 2000).

We consider possibly infinite games of perfect information in extensive form and fix the following sets:

- A set \mathcal{A} of **agents** or **players**. In our examples, we confine ourselves to two-player games, with $\mathcal{A} = \{A, B\}$.
- A set \mathcal{C} of **choices**. We restrict our discussion to games where a player has a choice of two options, *left* or *right*, at every stage in the game where it is their turn to play. Thus $\mathcal{C} = \{l, r\}$.
- A set \mathcal{U} of **utility functions**, which assign a utility to each player. We take $\mathcal{U} = \mathbb{R}^{\mathcal{A}}$.

The set of game trees is defined to be (the carrier of) the final coalgebra (\mathcal{G}, γ) of the functor

$$F_G : X \mapsto \mathcal{U} + \mathcal{A} \times X \times X.$$

on the category of **Set**. The game tree is a possibly infinite binary tree. The binary nodes are labelled with an agent whose turn it is to play at that stage in the game. The nodes have the form

$$\langle \alpha, g_l, g_r \rangle,$$

where α is the agent label, and g_l, g_r are the subgames corresponding to the left and right choices, respectively. The leaf nodes are labelled with utility functions, representing the pay-off for the game.

2.1. The 0/1 game

We define the utility functions to be

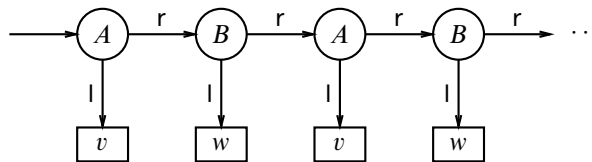
$$v := [A \mapsto 0, B \mapsto 1], \quad w := [A \mapsto 1, B \mapsto 0].$$

The game is defined by the following pair of mutually corecursive equations:

$$G = \langle A, v, H \rangle$$

$$H = \langle B, w, G \rangle.$$

We can depict the game tree as follows:



More formally, the above equations define an F_G -coalgebra $\alpha : \{G, H\} \rightarrow F_G\{G, H\}$ on the set $\{G, H\}$. The 0/1 game is given by $\llbracket G \rrbracket$, the image of G under the unique coalgebra morphism

$$\llbracket \cdot \rrbracket : (\{G, H\}, \alpha) \longrightarrow (G, \gamma)$$

from $(\{G, H\}, \alpha)$ to the final coalgebra.

Note that we can read the utility function v as ‘ A loses and B wins’, while w corresponds to ‘ A wins and B loses’. Thus the player who chooses to stop first loses.

2.2. The dollar auction

The 0/1 game has only finite states; it can be represented by a finite system of equations. The dollar auction game, a well-known example in game theory, is an infinite-state refinement, where the utility functions change as we move down the tree.

We fix a real number r , and define utility functions v_n, w_n for each stage $n \in \mathbb{N}$:

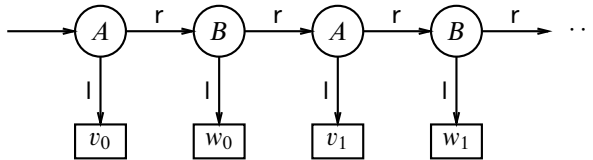
$$v_n := [A \mapsto -n, B \mapsto r - n], \quad w_n := [A \mapsto r - n, B \mapsto -n].$$

We define a game by simultaneous corecursion on the infinite set of variables $\{G_n, H_n \mid n \in \mathbb{N}\}$:

$$G_n = \langle A, v_n, H_n \rangle$$

$$H_n = \langle B, w_n, G_{n+1} \rangle.$$

The dollar auction game is again given by $\llbracket G_0 \rrbracket$, the image of G_0 under the unique coalgebra morphism into the final coalgebra. We can depict the game tree as follows:



The motivation behind this game is as follows (Shubik 1971). The value of the asset being bid for is r ; in the original example, r is one dollar or 100 cents. The asset goes to the highest bidder, who is left with a profit of $r - b$, where b is the value of his highest bid. The loser must also pay the value of his highest bid, while getting nothing in return. The above definition describes the situation where each player raises their bid by one cent at each stage in the game. A player either gives up and finishes the game, conceding the auction to the other player and accepting their loss, or continues, hoping that the other player will give up. Both players have an incentive to continue playing well beyond the point where both will make a loss, in order to try to minimize their losses.

3. Strategy profiles

Intuitively, a strategy for a player P of a game G specifies a choice (left or right) for every node of G at which it is P 's turn to move. A strategy profile specifies a strategy for every

player. Following (Lescanne 2011), we define the set of strategy profiles directly, as the final coalgebra (S, σ) of the functor

$$F_{SP} : X \mapsto \mathcal{U} + \mathcal{A} \times \mathcal{C} \times X \times X.$$

There is an evident natural transformation $t : F_{SP} \rightarrow F_G$ defined by projection, which induces a functor from the category of F_{SP} -coalgebras to the category of F_G -coalgebras. It sends a strategy profile $s \in S$ to the underlying game tree $\text{game}(s) \in \mathcal{G}$. We say that s is a strategy profile for the game G if $G = \text{game}(s)$.

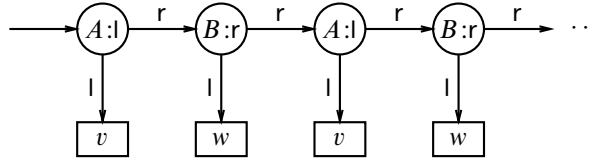
We now define a number of strategy profiles for the 0/1 and dollar auction games.

3.1. The 0/1 game

We define two strategy profiles for the 0/1 game where A always stops (plays left) and B always continues (plays right), by the following simultaneous corecursion:

$$\begin{aligned} \text{AsBc} &= \langle A, l, v, \text{BcAs} \rangle \\ \text{BcAs} &= \langle B, r, w, \text{AsBc} \rangle. \end{aligned}$$

We can depict this strategy profile as follows:



Formally, the strategy profile is given by $\llbracket \text{AsBc} \rrbracket$, the image of the strategy profile under the morphism to the final coalgebra. The strategy profile $\llbracket \text{AcBs} \rrbracket$, where A always continues and B always stops, is defined symmetrically.

3.2. The dollar auction

In a similar fashion, we define strategy profiles for the dollar auction by simultaneous corecursion on an infinite family of variables $\{\text{AsBc}_n, \text{BcAs}_n \mid n \in \mathbb{N}\}$:

$$\begin{aligned} \text{AsBc}_n &= \langle A, l, v_n, \text{BcAs}_n \rangle \\ \text{BcAs}_n &= \langle B, r, w_n, \text{AsBc}_{n+1} \rangle. \end{aligned}$$

Formally, $\llbracket \text{AsBc}_0 \rrbracket$ is a strategy profile for the dollar auction game in which A always stops and B always continues. The strategy profile $\llbracket \text{AcBs}_0 \rrbracket$ where A always continues and B always stops is defined symmetrically.

4. Subgame-perfect equilibrium

We now show, following Lescanne (2011), how the game-theoretic notion of **subgame-perfect equilibrium** can be defined coalgebraically. First, we introduce two auxiliary notions.

4.1. Weak and strong convergence

We introduce two predicates on strategy profiles, of **weak** and **strong convergence**. A strategy profile is weakly convergent if following the choices that it specifies from the root eventually leads to a leaf; it is strongly convergent if this holds in every sub-profile.

We define weak convergence by

$$s \downarrow \iff (s = U) \vee (s = \langle P, l, s_l, s_r \rangle \wedge s_l \downarrow) \vee (s = \langle P, r, s_l, s_r \rangle \wedge s_r \downarrow).$$

More formally, weak convergence is an element of the powerset $\mathcal{P}(S)$. It is defined as the least fixpoint of the monotone function

$$f_{\downarrow} : \mathcal{P}(S) \longrightarrow \mathcal{P}(S) :: S \mapsto \mathcal{U} \cup \{ \langle P, l, s_l, s_r \rangle \mid s_l \in S \} \cup \{ \langle P, r, s_l, s_r \rangle \mid s_r \in S \}.$$

Strong convergence is defined as follows:

$$s \Downarrow \iff (s = U) \vee (s = \langle P, c, s_l, s_r \rangle \wedge s \downarrow \wedge s_l \downarrow \wedge s_r \downarrow).$$

More formally, it is the greatest fixpoint of the monotone function f_{\Downarrow} defined analogously to f_{\downarrow} :

$$f_{\Downarrow} : \mathcal{P}(S) \longrightarrow \mathcal{P}(S) :: S \mapsto \mathcal{U} \cup \{ \langle P, l, s_l, s_r \rangle \mid s_l \in S, s_r \in S \}.$$

These fixpoints exist by the Knaster–Tarski fixed point theorem (Knaster 1928; Tarski 1955).

Note that weak convergence is an **inductive** notion; the profile must specify a finite path from the root to a leaf. Thus it is defined as a least fixpoint. Strong convergence, by contrast, expresses a constraint on all subtrees of an infinite tree, and hence is defined **coinductively**, as a greatest fixpoint. The examples of the strategy profiles described in the previous section are all strongly convergent, as we shall prove in Section 6.

The relationship between the two notions can be nicely characterized in terms of coalgebraic modal logic (Moss 1999; Rößiger 2000). We can define an ‘always’ modality \square as an operator on $\mathcal{P}(S)$:

$$s \models \square \phi \iff s \models \phi \wedge s = \langle P, c, s_l, s_r \rangle \Rightarrow s_l \models \square \phi \wedge s_r \models \square \phi.$$

This operator is defined coinductively as a greatest fixpoint. This modality generalizes straightforwardly to any polynomial functor, and can in fact be defined in a much more general way in the context of coalgebraic modal logic (Moss 1999; Rößiger 2000). If we write WC for the weak convergence predicate, and SC for the strong convergence predicate, we have the following:

Proposition 4.1. $SC = \square WC$.

4.2. Utility functions induced by strategy profiles

A strategy profile s induces a utility function \hat{s} . This function is defined as follows:

$$\hat{s} = \begin{cases} U, & s = U \\ \hat{s}_l, & s = \langle P, l, s_l, s_r \rangle \\ \hat{s}_r, & s = \langle P, r, s_l, s_r \rangle. \end{cases}$$

In general, this function may be partial; however, if s is weakly convergent, \hat{s} is always a well-defined total function in \mathcal{U} .

4.3. Subgame-perfect equilibria

We are now ready to define the notion of a strategy profile being a **subgame-perfect equilibrium**. Firstly, we define a predicate PE on strategy profiles:

$$\text{PE}(s) \iff s \downarrow \wedge (s = \langle P, l, s_l, s_r \rangle \Rightarrow \hat{s}_l(P) \geq \hat{s}_r(P)) \wedge (s = \langle P, r, s_l, s_r \rangle \Rightarrow \hat{s}_r(P) \geq \hat{s}_l(P)).$$

This predicate is defined explicitly in terms of previous notions; neither induction nor coinduction is used. It says that the choice at the root for player P results in a better pay-off for P than the other choice would have done. Note that this predicate implies in particular that strong convergence holds.

We now define the subgame-perfect equilibrium predicate SPE on \mathcal{S} , coinductively as a greatest fixpoint.

$$\text{SPE}(s) \iff \text{PE}(s) \wedge (s = \langle P, c, s_l, s_r \rangle \Rightarrow \text{SPE}(s_l) \wedge \text{SPE}(s_r)).$$

This says that the PE predicate holds at every node of the tree and we get the following analogue of Proposition 4.1.

Proposition 4.2. $\text{SPE} = \Box\text{PE}$.

The next step is to show that all the strategy profiles discussed in the previous section are in fact subgame-perfect equilibria. In order to do this, we will need to have an appropriate proof principle in place. This is the topic of the next section.

5. Predicate coinduction

The main emphasis in coinductive proofs has been on proving equations; the main tool for this is provided by the notion of bisimulation. However, as emphasized e.g. by Kozen (Kozen and Ruozzi 2007), the scope of coinductive methods is broader than this. In our case, we are interested in **predicates** (properties) rather than equations. In particular, we wish to show that various elements of the final coalgebra \mathcal{S} satisfy the SPE predicate.

We shall formulate a proof principle which is adequate to carry out these proofs. The principle is quite general, and applies to any **Set**-functor T which has a final coalgebra, and is equipped with a **predicate lifting** (Jacobs 2005), from which a \Box -modality can be defined. Thus it applies in particular to polynomial functors such as F_{SP} .

First, we need some notation. The final coalgebra of T is denoted (\mathcal{S}, σ) . Suppose we have a T -coalgebra (X, α) , which we think of as a corecursive system of equations on the set of variables X . We define a map

$$\bar{\alpha} : \mathcal{S}^X \longrightarrow \mathcal{S}^X :: \eta \mapsto [x \mapsto \sigma^{-1} \circ T\eta \circ \alpha(x)].$$

Here $\sigma^{-1} : T\mathcal{S} \longrightarrow \mathcal{S}$ is the inverse of σ , which is an isomorphism by the Lambek lemma (Lambek 1968).

The following proposition follows directly by unravelling the definitions and applying the final coalgebra property:

Proposition 5.1. The map $\bar{\alpha}$ has a unique fixpoint $\bar{\alpha}^* \in \mathcal{S}^X$; moreover, $\bar{\alpha}^* = \llbracket \cdot \rrbracket$, the unique coalgebra morphism from (X, α) to the final coalgebra.

Now let $\phi \subseteq \mathcal{S}$ be a predicate on \mathcal{S} . We can formulate our predicate coinduction principle as follows.

$$\frac{\forall \eta \in \mathcal{S}^X. \forall x \in X. \exists k \geq 1. \phi(\bar{\alpha}^k(\eta)(x))}{\forall x \in X. \Box \phi(\llbracket x \rrbracket)}$$

We shall show the soundness of this principle in the following proposition.

Proposition 5.2. The predicate coinduction principle is sound.

Proof. We can consider the rule as derived by composing two more basic rules. First, from the premise, we claim that we can derive the following:

$$\forall x \in X. \phi(\llbracket x \rrbracket).$$

This holds by taking $\eta = \llbracket \cdot \rrbracket$, and noting that, by Proposition 5.1, $\bar{\alpha}^k(\llbracket \cdot \rrbracket)(x) = \llbracket x \rrbracket$.

Since the image of a coalgebra morphism is a sub-coalgebra (Rutten 2000, Theorem 6.3), and hence an invariant (Jacobs 2005, Theorem 6.2.5), and since $\Box \phi$ is the largest invariant contained in ϕ (Jacobs 2005, Definition 6.3.1), the conclusion of the rule now follows. \square

Simple as it is, this rule is useful since it allows us to derive invariants for elements of the final coalgebra which are defined by arbitrary systems of corecursive equations. We shall now apply it to the task of showing that the strategy profiles we have defined are subgame-perfect equilibria.

6. Proving properties for corecursively defined strategy profiles

We shall now apply our predicate coinduction principle to show that the strategy profiles defined in Section 3 are strongly convergent and subgame-perfect equilibria.

6.1. The 0/1 game

We shall begin by explicitly computing the depth-three unfoldings of the corecursion variables AsBc and BcAs. In the notation of the predicate coinduction rule, we are

computing $\bar{\alpha}^k(\eta)(x)$ for $k = 3$ and $x = \text{AsBc}$, $x = \text{BcAs}$. A straightforward application of the definitions yields

$$\text{AsBc}^{(3)} = \langle A, l, v, \langle B, r, w, \langle A, l, v, ? \rangle \rangle \rangle, \quad \text{BcAs}^{(3)} = \langle B, r, w, \langle A, l, v, \langle B, r, w, ? \rangle \rangle \rangle.$$

Here we write $?$ for $\eta(x)$, since η is arbitrary and we have no information about this value.

A similar computation yields

$$\text{AcBs}^{(3)} = \langle A, r, v, \langle B, l, w, \langle A, r, v, ? \rangle \rangle \rangle, \quad \text{BsAc}^{(3)} = \langle B, l, w, \langle A, r, v, \langle B, l, w, ? \rangle \rangle \rangle.$$

Proposition 6.1. The strategy profiles $\llbracket \text{AcBs} \rrbracket$ and $\llbracket \text{AsBc} \rrbracket$ are strongly convergent.

Proof. We apply the predicate coinduction principle, with $k = 2$. Using the computations of $\text{AcBs}^{(3)}$ and $\text{AsBc}^{(3)}$ given above, we can compute directly that $\llbracket \text{AcBs} \rrbracket \downarrow$ and $\llbracket \text{BsAc} \rrbracket \downarrow$, with $\llbracket \widehat{\text{AcBs}} \rrbracket = \llbracket \widehat{\text{BsAc}} \rrbracket = w$, and $\llbracket \text{AsBc} \rrbracket \downarrow$ and $\llbracket \text{BcAs} \rrbracket \downarrow$, with $\llbracket \widehat{\text{AcBs}} \rrbracket = \llbracket \widehat{\text{BsAc}} \rrbracket = v$. We conclude that $\llbracket \text{AcBs} \rrbracket \downarrow$ and $\llbracket \text{AsBc} \rrbracket \downarrow$, as required. \square

Proposition 6.2. The strategy profiles $\llbracket \text{AcBs} \rrbracket$ and $\llbracket \text{AsBc} \rrbracket$ are subgame-perfect equilibria.

Proof. We apply the predicate coinduction principle, with $k = 3$. For AsBc , we must verify that $\text{AsBc}^{(3)}$ and $\text{BcAs}^{(3)}$ are PE. Using the computation of $\text{AsBc}^{(3)}$ and $\text{BcAs}^{(3)}$ given above, this reduces to verifying the inequalities

$$v(A) = 0 \geq 0 = v(A), \quad v(B) = 1 \geq 0 = w(B).$$

The verification that $\llbracket \text{AcBs} \rrbracket$ is SPE is similar. \square

6.2. The dollar auction

The analysis for the dollar auction will proceed along very similar lines to the 0/1 game.

We begin by computing the depth-three unfoldings of the corecursion variables AsBc_n and BcAs_n for all $n \geq 0$.

$$\text{AsBc}_n^{(3)} = \langle A, l, v_n, \langle B, r, w_n, \langle A, l, v_{n+1}, ? \rangle \rangle \rangle, \quad \text{BcAs}_n^{(3)} = \langle B, r, w_n, \langle A, l, v_{n+1}, \langle B, r, w_{n+1}, ? \rangle \rangle \rangle.$$

A similar computation yields

$$\text{AcBs}_n^{(3)} = \langle A, r, v_n, \langle B, l, w_n, \langle A, r, v_{n+1}, ? \rangle \rangle \rangle, \quad \text{BsAc}_n^{(3)} = \langle B, l, w_n, \langle A, r, v_{n+1}, \langle B, l, w_{n+1}, ? \rangle \rangle \rangle.$$

The following result can now be proved using predicate coinduction, with $k = 2$, just as for the 0/1 game.

Proposition 6.3. The strategy profiles $\llbracket \text{AcBs}_0 \rrbracket$ and $\llbracket \text{AsBc}_0 \rrbracket$ are strongly convergent.

We recall the parameter r used to define the utility functions v_n, w_n .

Proposition 6.4. If $r \geq 1$, the strategy profiles $\llbracket \text{AcBs}_n \rrbracket$ and $\llbracket \text{AsBc}_n \rrbracket$ are subgame-perfect equilibria for all n .

Proof. We apply the predicate coinduction principle, with $k = 3$. For AsBc_n , we must verify that $\text{AsBc}_n^{(3)}$ and $\text{BcAs}_n^{(3)}$ are PE for all n . Using the computations of $\text{AsBc}_n^{(3)}$ and

$\text{BcAs}_n^{(3)}$ given above, this reduces to verifying the inequalities

$$v_n(A) = -n \geq -(n + 1) = v_{n+1}(A), \quad v_{n+1}(B) = r - (n + 1) \geq -n = w_n(B).$$

The latter inequalities are satisfied if and only if $r \geq 1$. □

7. The one-deviation principle

We now verify an important property of subgame-perfect equilibria: a strategy profile is SPE if and only if it dominates any profile which differs from it in exactly one choice. In the standard game theoretical literature (Fudenberg and Levine 1983), this is proved for infinite games only under strong continuity assumptions on the pay-offs, which amount to discounting beyond some finite horizon, thus allowing reduction to standard backwards induction reasoning. We need no such assumptions.

Given a strategy profile s of the form $\langle P, c, s_l, s_r \rangle$, we say that a profile t for the same game is a one-deviation from s if t has one of the following forms:

- $\langle P, c', s_l, s_r \rangle$, where $c' \neq c$.
- $\langle P, c, s'_l, s_r \rangle$, where s'_l is a one-deviation from s_l .
- $\langle P, c, s_l, s'_r \rangle$, where s'_r is a one-deviation from s_r .

This is an inductive definition. Given a strategy profile s we define a relation $s \succcurlyeq t$, where t is a one-deviation of s , inductively as follows:

- If $s = \langle P, l, s_l, s_r \rangle$ and $t = \langle P, r, s_l, s_r \rangle$, then $s \succcurlyeq t$ iff $\hat{s}_l(P) \geq \hat{s}_r(P)$.
- If $s = \langle P, r, s_l, s_r \rangle$ and $t = \langle P, l, s_l, s_r \rangle$, then $s \succcurlyeq t$ iff $\hat{s}_r(P) \geq \hat{s}_l(P)$.
- If $s = \langle P, c, s_l, s_r \rangle$ and $t = \langle P, c, s'_l, s_r \rangle$, then $s \succcurlyeq t$ iff $s_l \succcurlyeq s'_l$.
- If $s = \langle P, c, s_l, s_r \rangle$ and $t = \langle P, c, s_l, s'_r \rangle$, then $s \succcurlyeq t$ iff $s_r \succcurlyeq s'_r$.

Proposition 7.1 (The one-deviation principle). A strongly convergent strategy profile s is SPE if and only if for every one-deviation t , $s \succcurlyeq t$.

Proof. First, note that if $\neg(s \succcurlyeq t)$ for some one-deviation t , this means that some sub-profile of s does not satisfy PE, and since $\text{SPE} = \square\text{PE}$, this implies that s does not satisfy SPE. For the converse, note that if for some one-deviation t , $s \succcurlyeq t$, this implies that the sub-profile of s whose root is at the node where t differs from s satisfies PE. If this holds for all one-deviations, then all sub-profiles of s satisfy PE, and hence s satisfies SPE. □

8. Complete characterization of SPE for the dollar auction

We can give a **complete characterization** of the subgame-perfect equilibria for the dollar auction game.

Theorem 8.1. A strategy profile for the dollar auction game is SPE if and only if it is of the form AsBc or *A always stops and B always continues* or symmetrically AcBs or *B always stops and A always continues*.

Proof. First, note that if both players always continue from some point in the game, the profile will not be strongly convergent. Now suppose that both players choose to stop at some nodes of the game. Then there must be some node v where player α chooses to stop, such that the next player choosing to stop is the other player. But then v is not in PE, since α could improve his pay-off from that node by choosing to continue. Thus one player must always continue in any SPE profile, while the other player α must stop infinitely often. Finally, α must in fact always choose to stop, since otherwise he could improve his pay-off from any node where he chooses to continue. \square

This analysis applies to any game sharing the following features of the dollar auction game:

- 1 At any point of the game, it is always better for a given player if the other player stops first.
- 2 At any point of the game, it is (strictly) better for the player who is the first to stop from that point to stop immediately rather than later.

As a final remark on the dollar auction game, we note that Lescanne proposes to use this form of analysis to explain the **rationality of infinite escalation** (Lescanne 2011). If we don't know which strategy the other player is following, we always have an incentive to continue! However, by our characterization result, after one round where both players choose to continue, **they both know they are not in an SPE** – and all bets are off! Thus it seems to us that a comprehensive analysis of escalation should use a refined model with an explicit representation of the beliefs of the players, as in Harsanyi type spaces (Harsanyi 1967) – which can also be modelled coalgebraically (Moss and Viglizzo 2004).

9. Further directions

The present paper contains what should be regarded as some very preliminary results, presented in a manner which is mainly aimed at computer scientists and mathematicians rather than economists and game theorists. Nevertheless, in our view the general idea of applying coalgebraic and other structural methods which have been developed in computer science to economics and game theory is promising, and deserves further study and development. In particular, many topics in economics which refer to infinite horizons and reflexivity seem tailor-made for the use of coalgebraic methods. At the same time, they can suggest new challenges and technical directions for coalgebra.

References

- Fudenberg, D. and Levine, D. (1983). Subgame-perfect equilibria of finite- and infinite-horizon games. *Journal of Economic Theory* **31** (2) 251–268.
- Fudenberg, D. and Tirole, J. (1991). *Game Theory*, MIT Press.
- Harsanyi, J. C. (1967). Games with incomplete information played by 'Bayesian' players, I-III. Part I. The basic model. *Management Science* **14** (3) 159–182.
- Honsell, F. and Lenisa, M. (2011). Conway games, algebraically and coalgebraically. *preprint arXiv:1107.1351*.

- Honsell, F., Lenisa, M. and Redamalla, R. (2012). Categories of coalgebraic games. In: *Proceedings of the 37th International Symposium, Mathematical Foundations of Computer Science*, Springer 503–515.
- Jacobs, B. (2005). Introduction to coalgebra. Towards Mathematics of States and Observations. Available at <http://www.cs.ru.nl/B.Jacobs/CLG/JacobsCoalgebraIntro.pdf>, 22.
- Jacobs, B. and Rutten, J. (1997). A tutorial on (co) algebras and (co) induction. *Bulletin-European Association for Theoretical Computer Science* **62** 222–259.
- Knaster, B. (1928). Un théoreme sur les fonctions d'ensembles. *Annales de la Société Polonaise de Mathématique* **6** (133) 2013134.
- Kozen, D. and Ruozzi, N. (2007). Applications of metric coinduction. In: *Proceedings of the 2nd International Conference on Algebra and Coalgebra in Computer Science*, Springer-Verlag 327–341.
- Lambek, J. (1968). A fixpoint theorem for complete categories. *Mathematische Zeitschrift* **103** (2) 151–161.
- Lescanne, P. (2011). Rationality and escalation in infinite extensive games. *preprint arXiv:1112.1185*.
- Lescanne, P. and Perrinel, M. (2012). “Backward” coinduction, Nash equilibrium and the rationality of escalation. *Acta Informatica* **49** (3) 117–137.
- Ljungqvist, L. and Sargent, T. J. (2004). *Recursive Macroeconomic Theory*, 2nd edition, MIT Press.
- Moss, L. (1999). Coalgebraic logic. *Annals of Pure and Applied Logic* **96** (1) 277–317.
- Moss, L. S. and Viglizzo, I. D. (2004). Harsanyi type spaces and final coalgebras constructed from satisfied theories. *Electronic Notes in Theoretical Computer Science* **106** 279–295.
- Rößiger, M. (2000). Coalgebras and modal logic. *Electronic Notes in Theoretical Computer Science* **33** 294–315.
- Rutten, J. (2000). Universal coalgebra: A theory of systems. *Theoretical Computer Science* **249** (1) 3–80.
- Shubik, M. (1971). The dollar auction game: A paradox in noncooperative behavior and escalation. *Journal of Conflict Resolution* 109–111.
- Tarski, A. (1955). A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics* **5** (2) 285–309.